Open Problems about Curves, Sets, and Measures

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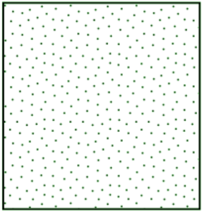
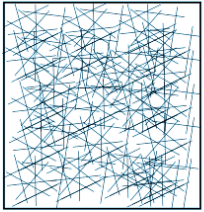
Preview: Structure of Measures

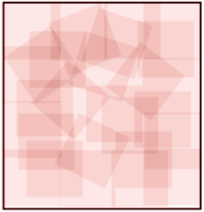
Three Measures. Let $a_i > 0$ be weights with $\sum_{i=1}^{\infty} a_i = 1$.
Let $\{x_i : i \geq 1\}$, $\{\ell_i : i \geq 1\}$, $\{S_i : i \geq 1\}$ be a dense set of points, unit line segments, unit squares in the plane.



$\mu_0 = \sum_{i=1}^{\infty} a_i \, \delta_{x_i}$



$\mu_1 = \sum_{i=1}^{\infty} a_i \, L^1|_{\ell_i}$



$\mu_2 = \sum_{i=1}^{\infty} a_i \, L^2|_{S_i}$

- μ_0, μ_1, μ_2 are probability measures on $\mathbb{R}^2$
- The support of μ is the smallest closed set carrying μ;
  spt $\mu_0 =$ spt $\mu_1 =$ spt $\mu_2 = \mathbb{R}^2$
- μ_i is carried by i-dimensional sets (points, lines, squares)
- **The support of a measure is a rough approximation that hides the underlying structure of a measure**

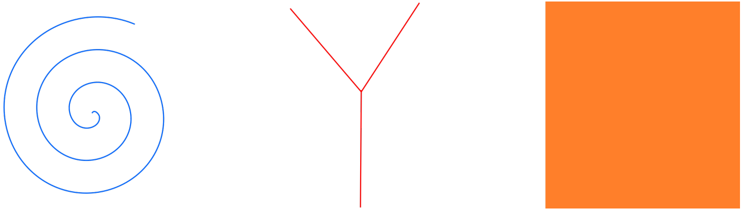
Part I. Curves

Part II. Subsets of Curves

Part III. Rectifiability of Measures

What is a curve?

A **curve** $\Gamma \subset \mathbb{R}^n$ is a **continuous image** of $[0, 1]$:
There exists a continuous map $f : [0, 1] \to \mathbb{R}^n$ such that $\Gamma = f([0, 1])$

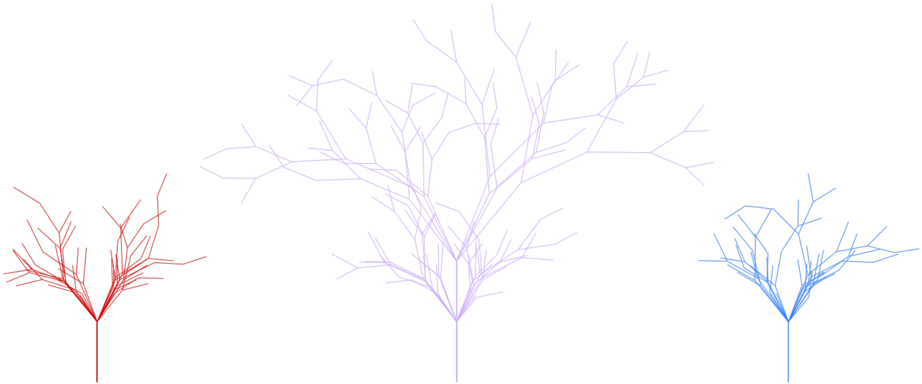


A continuous map f with $\Gamma = f([0, 1])$ is called a **parameterization** of Γ

- There are curves which do not have a 1-1 parameterization
- There are curves which have topological dimension > 1

A curve Γ is **rectifiable** if $\exists f$ with $\sup_{x_0 \leq \cdots \leq x_k} \sum_{j=1}^{k} |f(x_j) - f(x_{j-1})| < \infty$

When I think of curves...



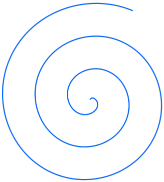
View from the UConn Math Department

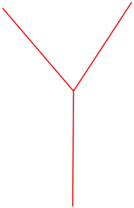
When is a set a curve?

Theorem (Hahn-Mazurkiewicz)

A nonempty set $\Gamma \subset \mathbb{R}^n$ *is a curve if and only if*

Γ *is compact, connected, and locally connected*





The proof of the forward direction is an exercise

The proof of the reverse direction is content of the theorem:
must **construct a parameterization** from only topological information

Examples of sets which are not curves

Theorem (Hahn-Mazurkiewicz)

A nonempty set $\Gamma \subset \mathbb{R}^n$ is not a curve if and only if

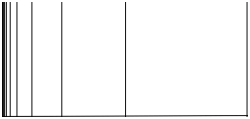
Γ is not compact or disconnected or not locally connected

Unbounded a straight line

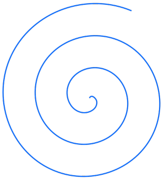
Not Closed an open line segment

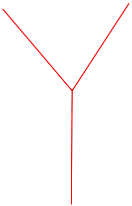
Disconnected a Cantor set

Not Locally Connected a comb



When is a set a rectifiable curve?





Theorem (Ważewski)

Let $\Gamma \subset \mathbb{R}^n$ be nonempty. TFAE:

1. *Γ is a rectifiable curve (finite total variation)*
2. *Γ is compact, connected, and $\mathcal{H}^1(\Gamma) < \infty$*
3. *Γ is a Lipschitz curve, i.e. there exists a Lipschitz continuous map $f : [0,1] \to \mathbb{R}^n$ such that $\Gamma = f([0,1])$*

$\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure

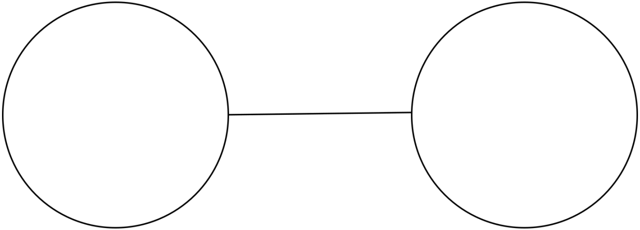
f is Lipschitz if $\exists C < \infty$ such that $|f(x) - f(y)| \leq C|x - y|$ for all x, y

The proof of $(1) \Rightarrow (2)$ is an exercise

The proof of $(3) \Rightarrow (1)$ is trivial

Proof by Picture

$\Gamma \subset \mathbb{R}^n$ is compact, connected, $\mathcal{H}^1(\Gamma) < \infty \implies \Gamma$ is Lipschitz curve



Goal: build a parameterization for the set Γ

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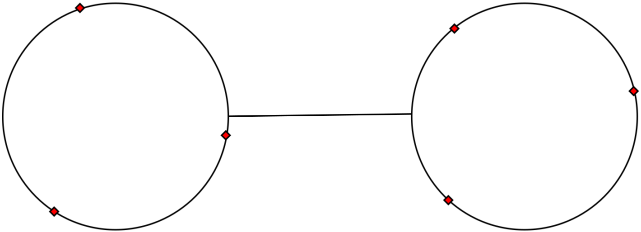

Step 1: approximate Γ by 2^{-k}-nets V_k, $k \geq 1$

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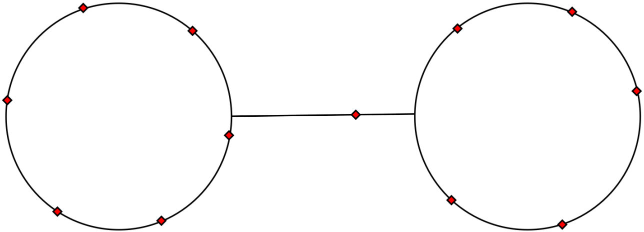

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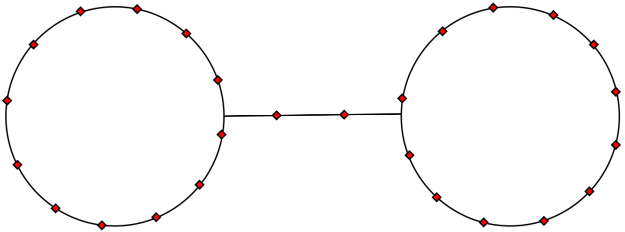

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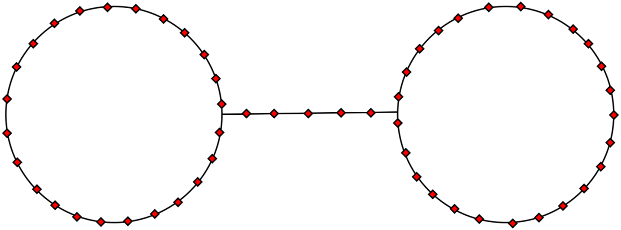

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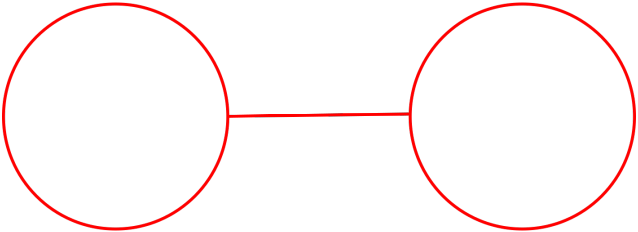

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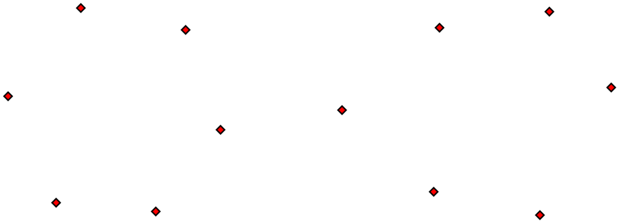

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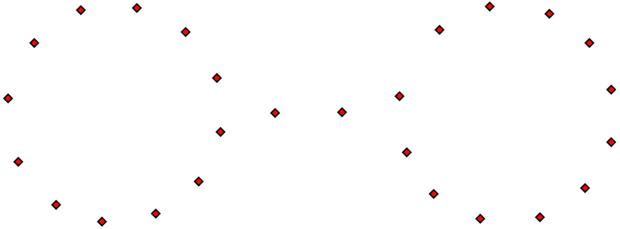

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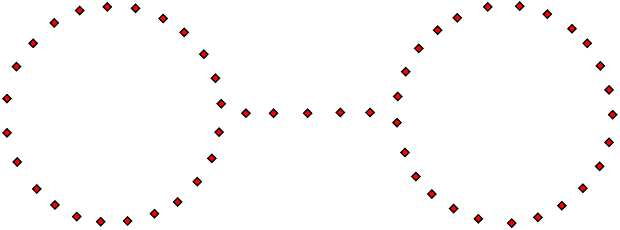

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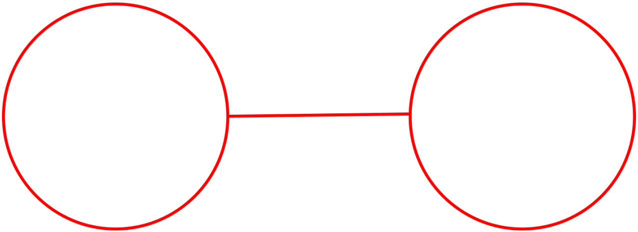

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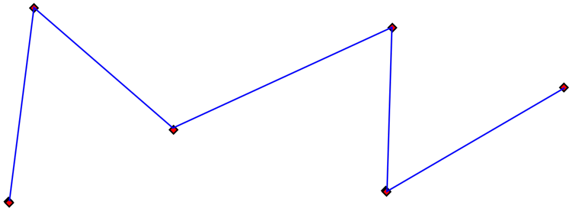

Step 2: draw piecewise linear spanning tree Γ_k through V_k

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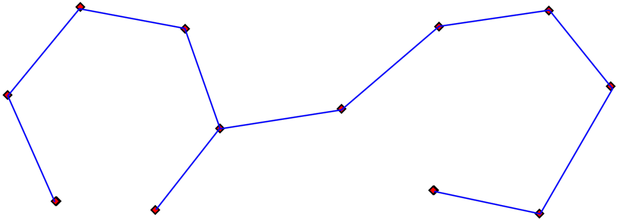

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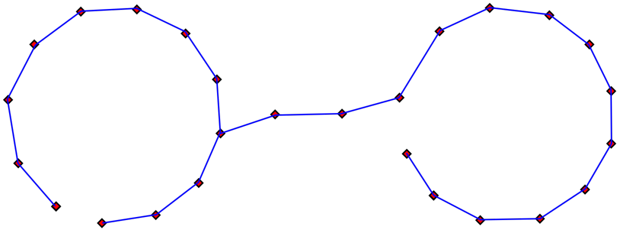

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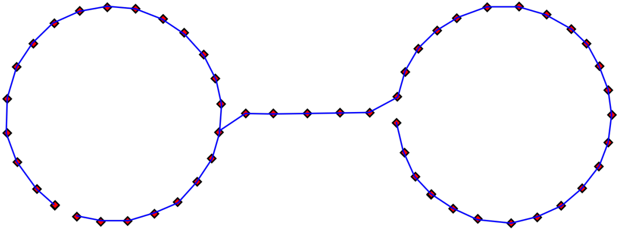

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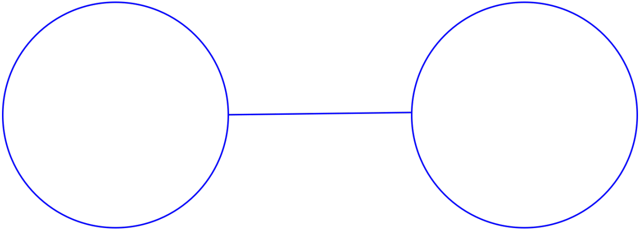

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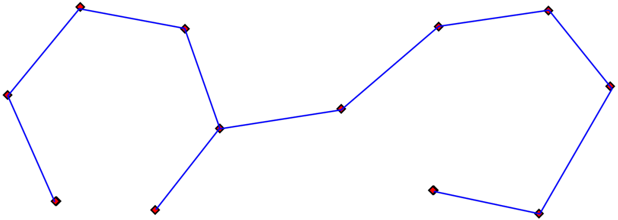

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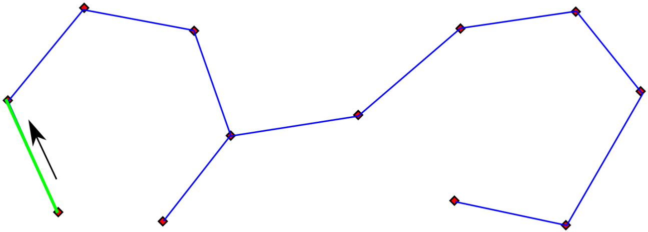

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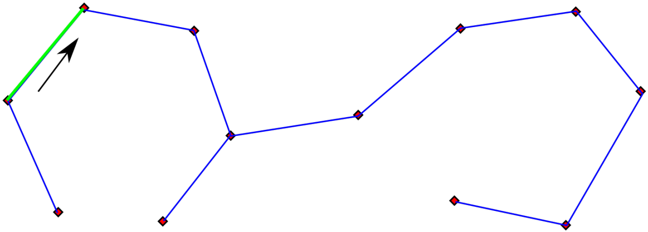

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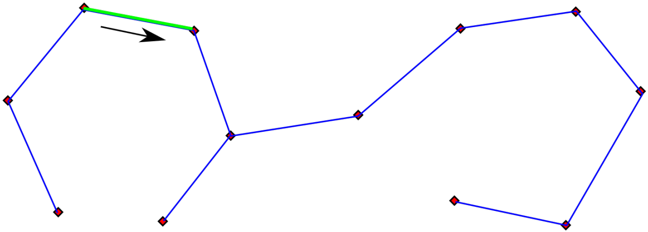

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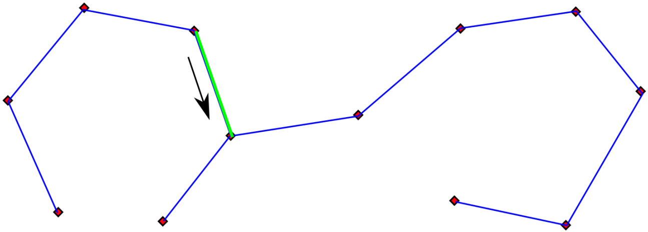

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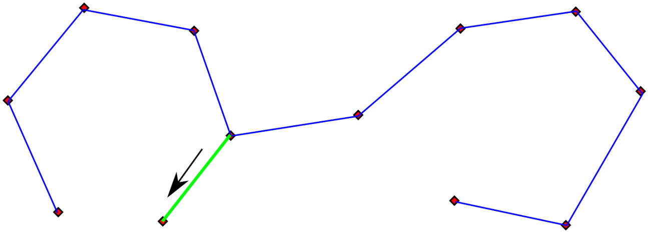

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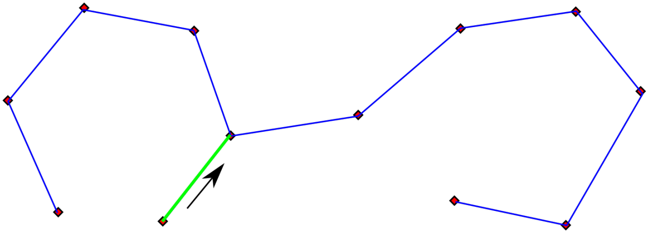

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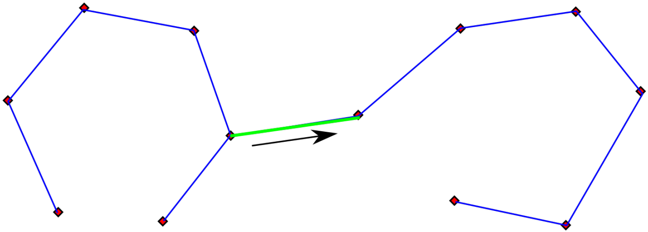

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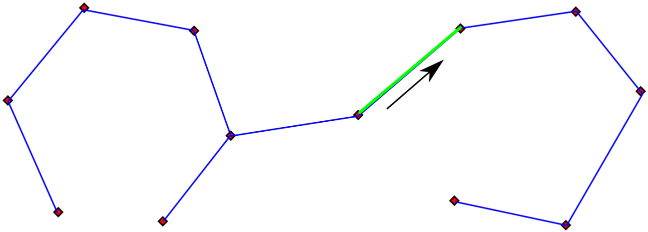

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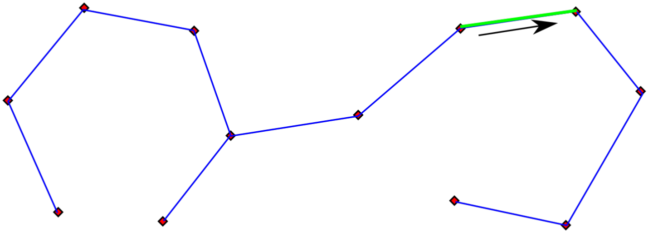

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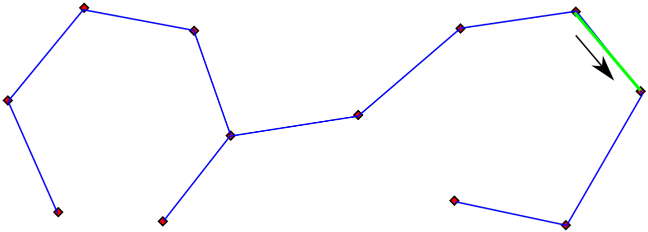

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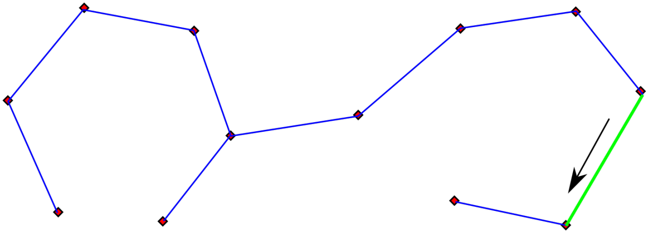

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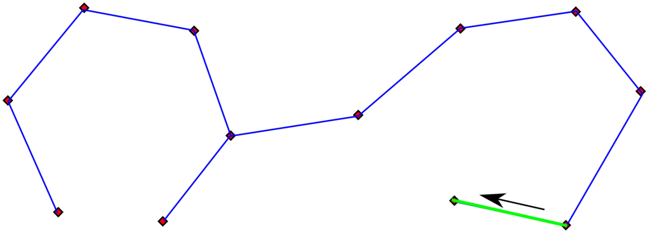

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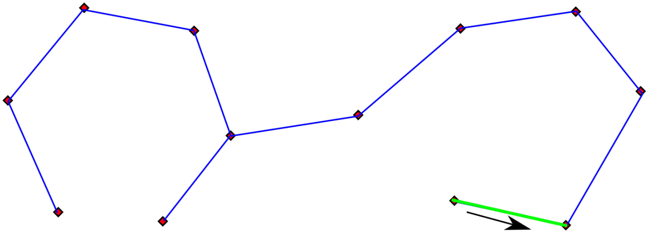

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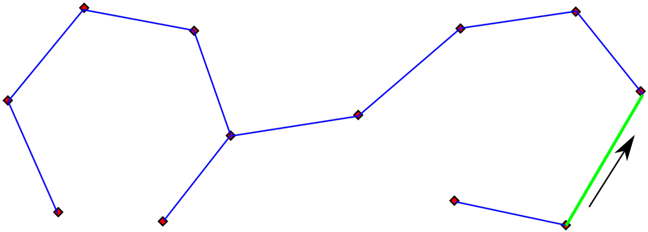

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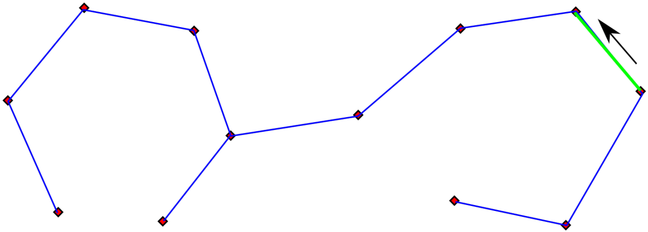

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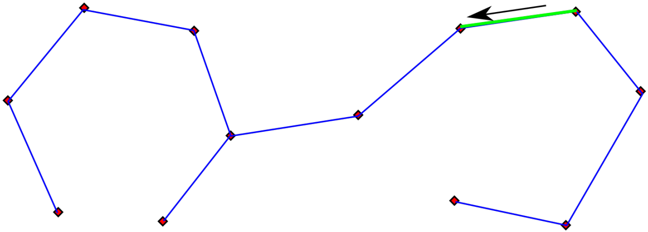

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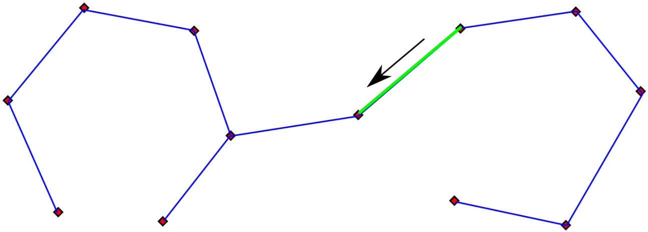

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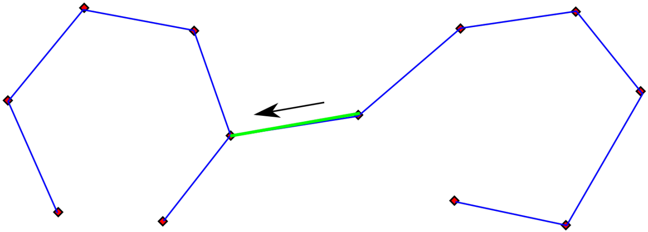

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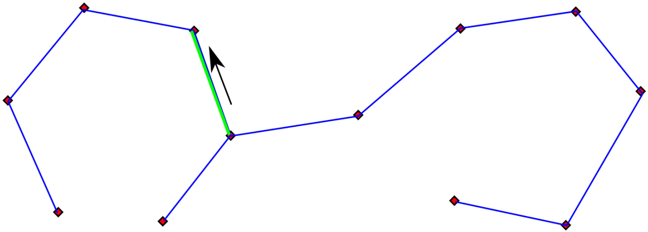

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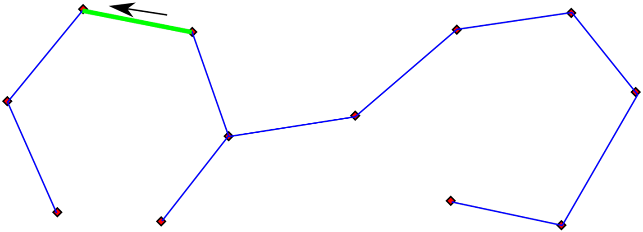
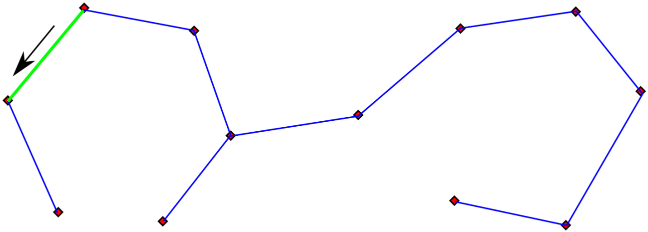


Step 3: pick a 2-1 tour of edges in the tree Γ_k

Proof by Picture

$\Gamma \subset \mathbb{R}^n$ is compact, connected, $\mathcal{H}^1(\Gamma) < \infty \implies \Gamma$ is Lipschitz curve



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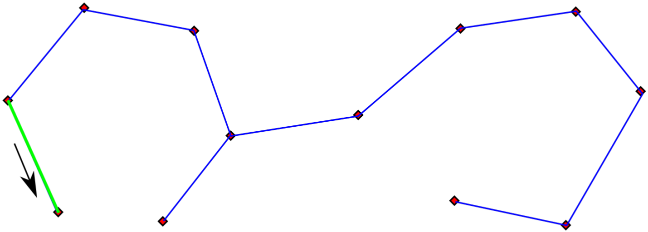

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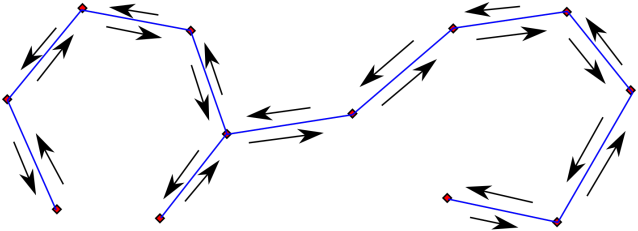

Step 4: tour defines piecewise linear map $f_k : [0, 1] \twoheadrightarrow \Gamma_k$

Proof by Picture

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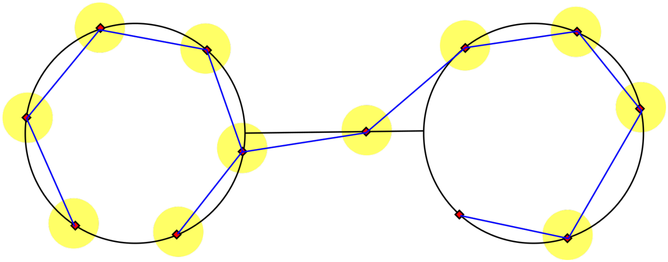
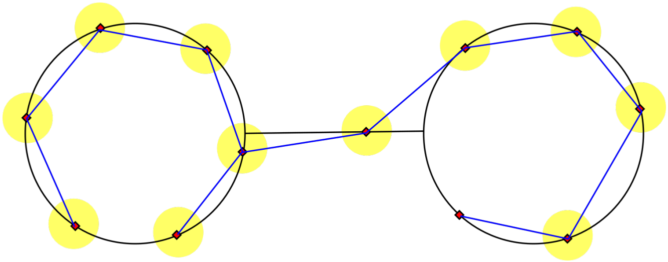


Step 5: length of i-th edge $\lesssim \mathcal{H}^1(E \cap B(v_i, \frac{1}{4} \cdot 2^{-k}))$

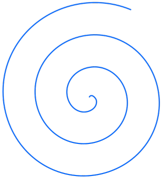
Proof by Picture

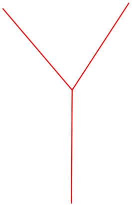
$\Gamma \subset \mathbb{R}^n$ is compact, connected, $\mathcal{H}^1(\Gamma) < \infty \implies \Gamma$ is Lipschitz curve



Conclusion: $\text{Lip}\, f_k \leq 32\mathcal{H}^1(\Gamma)$. Hence $f_{k_j} \rightrightarrows f : [0,1] \twoheadrightarrow \Gamma$ Lipschitz

Open Problem #1





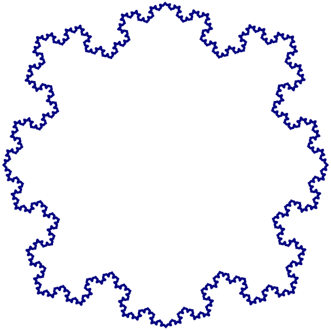
Theorem (Ważewski)

Let $\Gamma \subset \mathbb{R}^n$ be nonempty. TFAE:

1. *Γ is a rectifiable curve (finite total variation)*
2. *Γ is compact, connected, and $\mathcal{H}^1(\Gamma) < \infty$*
3. *Γ is a Lipschitz curve, i.e. there exists a Lipschitz continuous map $f : [0, 1] \to \mathbb{R}^n$ such that $\Gamma = f([0, 1])$*

Generalize Ważewski's theorem to higher dimensional curves

Snowflakes and Squares



Snowflakes and Squares

Open Problem (#2)

For each real $s \in (1, \infty)$*, characterize curves* $\Gamma \subset \mathbb{R}^n$ *with* $\mathcal{H}^s(\Gamma) < \infty$

Open Problem (#3)

For each real $s \in (1, \infty)$*, characterize* $(1/s)$***-Hölder curves****, i.e. sets that can be presented as* $h([0, 1])$ *for some map* $h : [0, 1] \to \mathbb{R}^n$ *with*

$$|h(x) - h(y)| \leq C|x - y|^{1/s}$$

Open Problem (#4)

For each integer $m \geq 2$*, characterize* ***Lipschitz*** m***-cubes****, i.e. sets that can be presented as* $f([0, 1]^m)$ *for some Lipschitz map* $f : [0, 1]^m \to \mathbb{R}^n$*.*

Obstruction to a Hölder Ważewski Theorem

- Every $(1/s)$-Hölder curve has $\mathcal{H}^s(\Gamma) < \infty$
- There are curves Γ with $\mathcal{H}^s(\Gamma) < \infty$ that are not $(1/s)$-Hölder.

Theorem (B, Naples, Vellis 2018)

*For all $s > 1$, there exists a curve $\Gamma \subset \mathbb{R}^n$ such that $\mathcal{H}^s(\Gamma \cap B(x, r)) \sim r^s$, but Γ is **not** a $(1/s)$-Hölder curve.*

Idea.

Look at the cylinder $C \times [0, 1] \subset \mathbb{R}^2$ over the standard "middle thirds" Cantor set $C \subset \mathbb{R}$. Adjoining the line segment $[0, 1] \times \{0\}$ makes the set connected, but it is not locally connected. Adjoining additional intervals $I_i \times \{t_j\}$ on a dense set of heights ("rungs") makes the set locally connected. We call this a **Cantor ladder**.

A modified version of this gives the desired set. □

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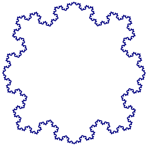
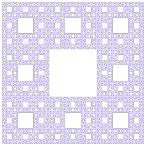
Sufficient conditions for Hölder curves





Theorem (Remes 1998)

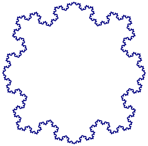
Let $S \subset \mathbb{R}^n$ be a ***self-similar set*** *satisfying the open set condition. If S is connected, then S is a $(1/s)$-Hölder curve, $s = \dim_H S$.*

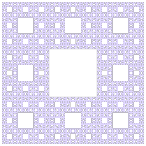
A set $E \subset \mathbb{R}^n$ is ε-**flat** if for every $x \in E$ and $0 < r \leq \operatorname{diam} E$, there exists a line ℓ such that $\operatorname{dist}(x, \ell) \leq \varepsilon r$ for all $x \in E \cap B(x, r)$.

Theorem (B, Naples, Vellis 2018)

Assume that $E \subset \mathbb{R}^n$ is ε-flat with $\varepsilon \ll 1$. If E is connected, compact, $\mathcal{H}^s(E) < \infty$ and $\mathcal{H}^s(E \cap B(x, r)) \gtrsim r^s$, then E is a $(1/s)$-Hölder curve with a one-to-one parameterization.

Sufficient conditions for Hölder curves





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Part I. Curves

Part II. Subsets of Curves

Part III. Rectifiability of Measures

Analyst's Traveling Salesman Problem

Given a bounded set $E \subset \mathbb{R}^n$ (an infinite list of cities), decide whether or not E is a subset of a rectifiable curve.

If so, construct a rectifiable curve Γ containing E that is short as possible.

This is solved for

- E in $\mathbb{R}^2$ by P. Jones (1990)
- E in $\mathbb{R}^n$ by K. Okikiolu (1992)
- E in ℓ_2 by R. Schul (2007)
- E in first Heisenberg group $\mathbb{H}^1$ by S. Li and R. Schul (2016)
- E in Laakso-type spaces by G.C. David and R. Schul (2017)
- E is Carnot group by V. Chousionis, S. Li, S. Zimmerman (2018): necessary condition only

Not contained in a rectifiable curve: a countable compact set with one accumulation point

For each $k \geq 2$, choose $m_k = k^2$ so that $\sum_{k=2}^{\infty} m_k^{-1} < \infty$. Arrange squares S_k with side length m_k^{-1} so that one side of each square lies on a given line; separate S_k and S_{k+1} by distance m_k^{-1}. Let V_k be collection of m_k^2 points in S_k separated by distance at least m_k^{-2}. **Let E be the closure of $\overline{\bigcup_{k=2}^{\infty} V_k}$.**

Suppose $\Gamma = f([0,1]) \supset E$ for some f with $|x - y| \geq L^{-1}|f(x) - f(y)|$

To contain V_k, the curve Γ must cross $m_k^2 - 1$ gaps of length at least m_k^{-2}. Requires at least $\frac{1}{2}L^{-1}$ of length in the domain of f by Lipschitz condition.

So for Γ to contain E there would have to be infinite length in the domain of f, which is a contradiction.

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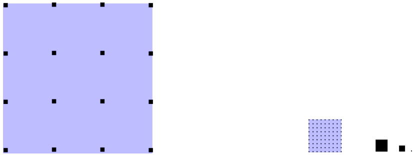

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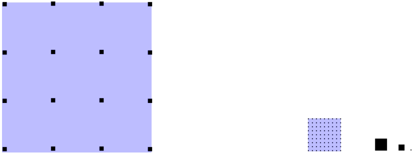

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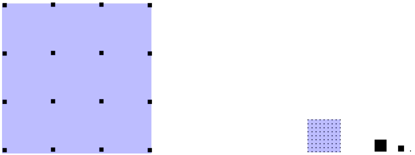

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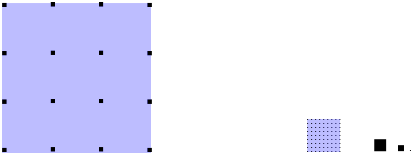

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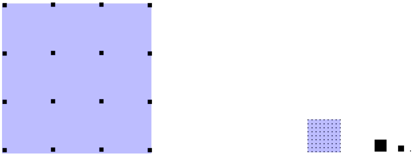

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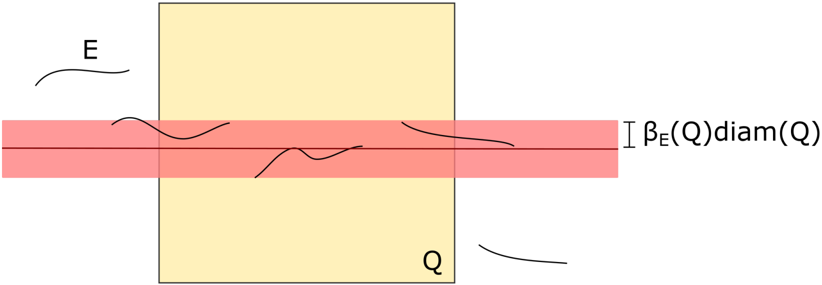
Unilateral Linear Approximation Numbers

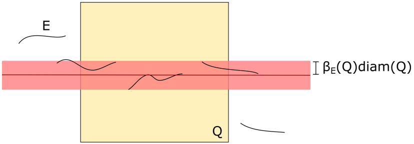


For any nonempty set $E \subset \mathbb{R}^n$ and bounded "window" $Q \subset \mathbb{R}^n$, the **Jones beta number** of E in Q is

$$\beta_E(Q) := \inf_{\text{line } \ell} \sup_{x \in E \cap Q} \frac{\operatorname{dist}(x, \ell)}{\operatorname{diam} Q} \in [0, 1].$$

If $E \cap Q = \emptyset$, we also define $\beta_E(Q) = 0$.

Analyst's Traveling Salesman Theorem



Theorem (P. Jones (1990), K. Okikiolu (1992))

Let $E \subset \mathbb{R}^n$ be a bounded set. Then E is contained in a rectifiable curve if and only if

$$S_E := \sum_{\text{dyadic } Q} \beta_E(3Q)^2 \operatorname{diam} Q < \infty$$

More precisely:

1. *If $S_E < \infty$, then there is a curve $\Gamma \supset E$ such that $\mathcal{H}^1(\Gamma) \lesssim_n \operatorname{diam} E + S_E$.*
2. *If Γ is a curve containing E, then $\operatorname{diam} E + S_E \lesssim_n \mathcal{H}^1(\Gamma)$.*

Open Problem #5

Theorem (P. Jones (1990), K. Okikiolu (1992))

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2. *If Γ is a curve containing E, then* $\operatorname{diam} E + S_E \lesssim_n \mathcal{H}^1(\Gamma)$.

Find characterizations of subsets of other nice families of sets

Hölder Traveling Salesman Theorem

Theorem (B, Naples, Vellis 2018)

For all $s > 1$, there exists a constant $\beta_0 = \beta_0(s, n) > 0$ such that:
If $E \subset \mathbb{R}^n$ is a bounded set and

$$\sum_{\substack{Q \text{ dyadic} \\ \beta_E(3Q) \geq \beta_0}} (\text{diam } Q)^s < \infty,$$

then E is contained in a $(1/s)$-Hölder curve.

Corollary

Assume $s > 1$. If $E \subset \mathbb{R}^n$ is a bounded set and

$$\sum_{\substack{Q \text{ dyadic} \\ \text{side} Q \leq 1}} \beta_E(3Q)^2 (\text{diam } Q)^s < \infty,$$

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Hölder Traveling Salesman Theorem

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Remarks

- There is a version of the theorem in infinite-dimensional Hilbert space
- Construction of approximating curves Γ_k are similar to case $s = 1$
- But unlike the case $s = 1$, we do not have Ważewski's theorem!!!
- So we have reimagine Jones' proof of the traveling salesman construction and build explicit parameterization of the Γ_k
- The condition is not necessary (e.g. fails for a Sierpinski carpet)

Part I. Curves

Part II. Subsets of Curves

Part III. Rectifiability of Measures

Measure Theorist's Traveling Salesman Problem

Given a finite Borel measure μ on $\mathbb{R}^n$ with bounded support ($\Leftrightarrow \mu(\mathbb{R}^n \setminus B) = 0$ for some bounded set B), decide whether or not μ is carried by a rectifiable curve.

If so, construct a rectifiable curve Γ carrying μ, i.e. $\mu(\mathbb{R}^n \setminus \Gamma) = 0$.

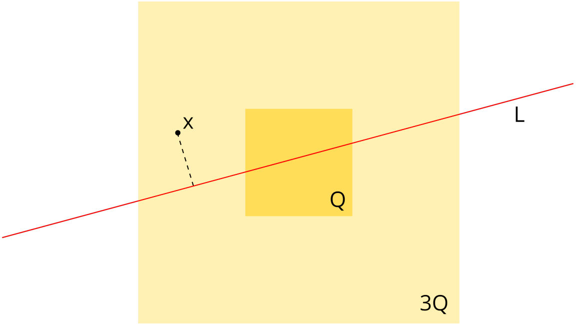
This is solved for

- μ such that $\mu(B(x, r)) \sim r$ for $x \in \operatorname{spt} \mu$ by Lerman (2003)
- μ any finite Borel measure by B and Schul (2017)

Non-homogeneous L^2 Jones β numbers

Let μ be a Radon measure on $\mathbb{R}^n$. For every cube Q, define $\beta_2(\mu, 3Q) = \inf_{\text{line } L} \beta_2(\mu, 3Q, L) \in [0, 1]$, where

$$\beta_2(\mu, 3Q, L)^2 = \int_{3Q} \left(\frac{\operatorname{dist}(x, L)}{\operatorname{diam} 3Q} \right)^2 \frac{d\mu(x)}{\mu(3Q)}$$



“Non-homogeneous” refers to the normalization $1/\mu(3Q)$.

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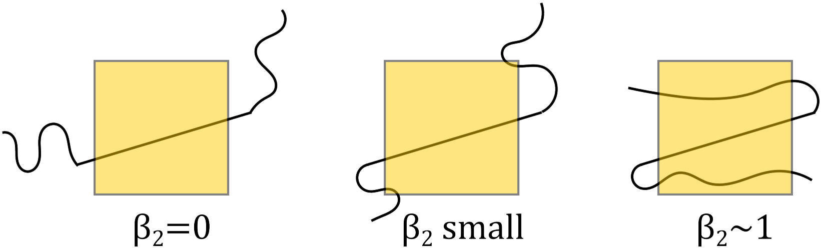

Traveling Salesman for Ahlfors Regular Measures

Theorem (Lerman 2003)

Let μ be a finite measure on $\mathbb{R}^n$ with bounded support. Assume that

$$\mu(B(x,r)) \sim r \quad \text{for all } x \in \operatorname{spt}\mu \text{ and } 0 < r \leq 1.$$

Then there is a rectifiable curve Γ such that $\mu(\mathbb{R}^n \setminus \Gamma) = 0$ if and only if

$$\sum_{\text{dyadic } Q} \beta_2(\mu, 3Q)^2 \operatorname{diam} Q < \infty.$$

Theorem (Martikainen and Orponen 2018)

There exists a Borel probability ν on $\mathbb{R}^2$ with bounded support such that

$$\sum_{\text{dyadic } Q} \beta_2(\nu, 3Q)^2 \operatorname{diam} Q < \infty$$

but ν is purely 1-unrectifiable, i.e. $\nu(\Gamma) = 0$ for every rectifiable curve Γ.

Traveling Salesman for Ahlfors Regular Measures

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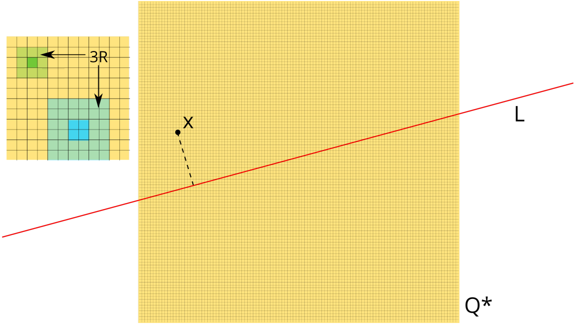
There exists a Borel probability ν on $\mathbb{R}^2$ with bounded support such that

$$\sum_{\text{dyadic } Q} \beta_2(\nu, 3Q)^2 \operatorname{diam} Q < \infty$$

but ν is purely 1-unrectifiable, i.e. $\nu(\Gamma) = 0$ for every rectifiable curve Γ.

Anisotropic L^2 Jones β numbers (B-Schul 2017)

Given dyadic cube Q in $\mathbb{R}^n$, $\Delta^*(Q)$ denotes a subdivision of $Q^* = 1600\sqrt{n}Q$ into dyadic cubes R of same / previous generation as Q s.t. $3R \subseteq Q^*$.



For every Radon measure μ on $\mathbb{R}^n$ and every dyadic cube Q, we define $\beta_2^{**}(\mu, Q)^2 = \inf_{\text{line } L} \max_{R\in\Delta^*(Q)} \beta_2(\mu, 3R, L)^2$, where

$$\beta_2(\mu, 3R, L)^2 = \int_{3R} \left(\frac{\text{dist}(x, L)}{\text{diam } 3R}\right)^2 \frac{d\mu(x)}{\mu(3R)}$$

Traveling Salesman Theorem for Measures

Theorem (B and Schul 2017)

Let μ be a finite measure on $\mathbb{R}^n$ with bounded support. Then there is a rectifiable curve Γ such that $\mu(\mathbb{R}^n \setminus \Gamma) = 0$ if and only if

$$\sum_{\text{dyadic } Q} \beta_2^{**}(\mu, Q)^2 \operatorname{diam} Q < \infty.$$

- Proof uses both halves of the traveling salesman theorem curves
- For the sufficient half, need extension of the traveling salesman construction without requirement $V_{k+1} \supset V_k$ (see B-Schul 2017)
- Using similar techniques, we can also get a characterization of countably 1-rectifiable Radon measures

Identification of 1-rectifiable Radon measures

For any Radon measure μ on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, the **lower density** is:

$$\underline{D}^1(\mu, x) \equiv \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{2r} \in [0, \infty]$$

and the **anisotropic square function** is:

$$J_2^*(\mu, x) \equiv \sum_{\substack{Q \text{ dyadic} \\ \text{diam } Q \leq 1}} \beta_2^*(\mu, Q)^2 \frac{\text{diam } Q}{\mu(Q)} \chi_Q(x) \in [0, \infty]$$

Theorem (B and Schul 2017)

If μ is a Radon measure on $\mathbb{R}^n$, then

- $\mu \llcorner \{x : \underline{D}^1(\mu, x) > 0$ *and* $J_2^*(\mu, x) < \infty\}$ *is countably 1-rectifiable*
- $\mu \llcorner \{x : \underline{D}^1(\mu, x) = 0$ *or* $J_2^*(\mu, x) = \infty\}$ *is purely 1-unrectifiable*

Open Problem #6

Given a measurable space $(X, \mathcal{M})$ and a family of sets $\mathcal{N}$, every σ-finite measure μ on $\mathbb{R}^n$ decomposes as $\mu = \mu_{\mathcal{N}} + \mu_{\mathcal{N}}^{\perp}$, where

- $\mu_{\mathcal{N}}$ is carried by $\mathcal{N}$: $\mu_{\mathcal{N}}(X \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0$ for some $\Gamma_i \in \mathcal{N}$
- $\mu_{\mathcal{N}}^{\perp}$ is singular to $\mathcal{N}$: $\mu_{\mathcal{N}}^{\perp}(\Gamma) = 0$ for all $\Gamma \in \mathcal{N}$.

Identification Problem:

Given $(X, \mathcal{M})$, $\mathcal{N} \subset \mathcal{M}$, and of $\mathcal{F}$ a family of σ-finite measures on $\mathcal{M}$, find properties $P(\mu, x)$ and $Q(\mu, x)$ defined for all $\mu \in \mathcal{F}$ and $x \in X$ such that

$\mu_{\mathcal{N}} = \mu \llcorner \{x : P(\mu, x)\}$ and $\mu_{\mathcal{N}}^{\perp} = \mu \llcorner \{x : Q(\mu, x)\}$

An important case is $X = \mathbb{R}^n$, $\mathcal{N}$ is Lipschitz images of $\mathbb{R}^m$ $(m \geq 2)$, and $\mathcal{F}$ is Radon measures on $\mathbb{R}^n$

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Criteria for fractional rectifiability

A model for **fractional rectifiability** based on Hölder continuous images of $\mathbb{R}^m$ in $\mathbb{R}^n$ was proposed by Martín and Mattila (1993,2000).

Theorem (B, Vellis 2018)

Let $s > 1$ and $m \leq t < s$. Assume that μ is a Radon measure on $\mathbb{R}^n$ such that

$$0 < \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{r^t} \leq \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^t} < \infty \quad \mu\text{-a.e. } x.$$

Then μ is carried by (m/s)-Hölder continuous images of $[0,1]^m$.

Theorem (B, Naples, Vellis 2018)

Let $s > 1$. Assume that μ is a Radon measure on $\mathbb{R}^n$ such that

$$\limsup_{r \downarrow 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty \quad \mu\text{-a.e. } x, \quad \text{and}$$

$$\int_0^1 \beta_2(\mu, B(x,r))^\alpha \frac{r^s}{\mu(B(x,r))} \frac{dr}{r} < \infty \quad \mu\text{-a.e. } x.$$

Then μ is carried by $(1/s)$-Hölder curves.

Thank you for listening!